FUNDAMENTAL CONCEPTS IN FILTER DESIGN

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2.1. Filters: concept and specifications

An electric filter is a system, or circuit, that modifies, distorts, deforms or manipulates in any form, the frequency spectrum of a signal —the filter’s input. This is realized according to a set of requirements, commonly known as specifications. Filters can be employed to attenuate or amplify the components of the input that are within a certain frequency range. They can be used also to reject or isolate specific frequency components. A filter can thus be considered to be a transmission system with the ability to let some frequency components pass and and some others not. In these conditions, the following frequency bands can be defined:

a) Passband (PB): is the frequency range, or ranges, for which input components are transmitted to the output. Each component of the input falling within this set of frequencies, is going to show up at the output port. Of course, not without some modification of its amplitude and phase.

b) Stopband (SB) or rejection band: is the frequency range, for which the input components are not transmitted to the input. Any input component at this set of frequencies is rejected, or at least greatly attenuated.

c) Transition band: frequency range between the passband and the stopband.

The specifications of the filter will consist in:

a) Cutoff and stopband frequencies: are the frequencies where the passband ends and the stopband begins, respectively.

b) Passband(s) ripple(s) and stopband(s) attenuation(s): are the attenuation limits beyond which the filter does not perform as desired. There will be a maximum attenuation allowed at each passband, the so-called ripple. There will be also a minimum required attenuation at each stopband.

c) Other characteristics related with the form of the transfer function: magnitude, phase, group delay, transmission zeroes, etc.
2.2. Filters classification

Lowpass filter

The basic function of a lowpass (LP) filter is to pass low frequencies from dc to some desired cutoff frequency and to attenuate high frequencies. The specifications for a lowpass filter are shown in Fig. 2.1. The LP filter passband is defined between 0 and the cutoff frequency. Signals within this range will be attenuated as much as dB. The stopband starts and ends at infinity. Input components at frequencies above must be attenuated dB at least. The frequency band within and is the transition band. The specifications of the LP filter are completely described by the parameters , , , and .

A second-order transfer function implementing a lowpass characteristic has the form:

\[
H(s) = \frac{V_o}{V_i} = \frac{b}{s^2 + as + b} = \frac{\omega_p^2}{s^2 + \omega_p^2 Q s + \omega_p^2} \quad (2.1)
\]

Attenuation at low frequencies approaches unity, 0dB, as shown in Fig. 2.2. While at higher frequencies it increases with \( s^2 \), in other words, with a slope of 40dB/decade. This function has two poles, as depicted in Fig. 2.2(b). The location of the poles establishes the shape of the filter response in the passband. For a high \( Q \), the peak in the passband ripple occurs approximately at \( \omega_p \). As \( Q \) grows, the peak will be larger and narrower. A common application: tone control in Hi-Fi amplifiers.

Highpass filters

A highpass (HP) filter passes all components at frequencies above the cutoff frequency \( \omega_p \), while attenuates all frequencies below the stopband border at \( \omega_s \). The passband spans from \( \omega_p \) to \( \infty \), while the stopband ranges from 0 to \( \omega_s \), as shown in Fig. 2.3. The highpass filter specifications are completely characterized by the parameters \( \omega_s \), \( \omega_p \), \( A_p \) and \( A_s \).

A second-order transfer function with a highpass characteristic is:

\[
\frac{V_o}{V_i} = \frac{s^2}{s^2 + \omega_p^2 + \omega_s^2 Q s + \omega_p^2 Q} \quad (2.2)
\]

This gain has a pair of complex conjugate poles, an a double zero at \( s = 0 \), as shown in Fig. 2.4(a). The attenuation at high frequencies approaches unity, as shown in Fig. 2.4(b).
Bandpass filters

A bandpass (BP) filter only pass components at frequencies within a particular band. Any component at a frequency outside this band is rejected, as shown in Fig. 2.5. The passband, that goes from $\omega_p$ to $\omega_s$, has a maximum attenuation of $A_p$. The rejection bands, ranging from DC to $\omega_s$ and from $\omega_p$ to $\infty$, have a minimum attenuation of $A_s$ dB.

A second order transfer function with a bandpass characteristic is:

$$V_o = \frac{\omega_o A_s}{s^2 + \omega_o^2 s + \omega_o^2}$$

(2.3)

Band reject filters

Band reject (BR) filters are employed to a particular band of frequencies, as shown in Fig. 2.7. The stopband extents from $\omega_{s1}$ to $\omega_{s2}$. The lower passband ranges from 0 to $\omega_p$, and the upper passband from $\omega_p$ to $\infty$. A second order function with band reject characteristic is:

This function has a pair of complex poles in the left-half-plane of the s-plane. It has also a zero at the origin and a zero at infinity (Fig. 2.6(a)). Both at low and high frequencies the attenuation increases like $s$, i.e. 20dB/decade. At the pole frequency, $\omega_o$, attenuation is one (Fig. 2.6(b)).
This function has complex poles in the left-half-plane of the $s$-plane. It has complex zeros in the $j\omega$-axis. If $\omega_z = \omega_o$, the frequency of the zeros equals the pole frequency (Fig. 2.8(a)). Attenuation at lower and higher frequencies approaches 1. At the zero, $s = j\omega_z$, attenuation reaches infinity (Fig. 2.8(b)). This kind of BR filter is referred as "symmetrical notch".

If $\omega_z > \omega_o$, the function in (2.4) represents a LP function with a transmission zero in the stopband (Fig. 2.9(a)). In this case, the attenuation at higher frequencies is larger than the attenuation at lower frequencies. This is known as a "low-pass-notch" filter. If $\omega_z < \omega_o$, the opposite occurs. Attenuation is larger at lower frequencies than at higher frequencies. The resulting function is a highpass filter with a transmission cero in the stopband. It is referred as "high-pass-notch" filter.

\[
\frac{V_o}{V_i} = \frac{s^2 + \omega_z^2}{s^2 + \frac{\omega_o^2}{Q}s + \omega_o^2}
\]

(2.4)
Delay equalizers

All the previous filter types have been defined attending to their magnitude of gain characteristics. Nothing has been said about their phase. In order to efficiently realize the magnitude specifications all the transmission zeros has been located at the \( j\omega \)-axis. The resulting transfer function has a minimum phase. If the delay has to be modified, a delay equalizer is employed after the filter. A delay equalizer permits compensating the distortion in the delay introduced by the previous filtering stage. It introduces a modification on the delay that makes the total delay (the sum of the delay introduced by the filter and the delay introduced by the equalizer) as plain as possible. Of course, the delay equalizer must not distort the attenuation of the filter.

Then, when magnitude and phase are specified, the transfer function can not be of minimum phase. This is, \( H(s) \) must have zeros in the right-half-plane. We can decompose this function into two components:

\[
H_{NMP}(s) = H_{MP}(s)H_{AP}(s)
\]

(2.5)

where \( NMP \) stands for “non-minimum phase”, \( MP \) for “minimum phase” and \( AP \) for “all-pass”. This decomposition is built by adding poles and zeros to \( H_{NMP}(s) \) at the mirrored position of the zeros. Then \( H_{AP}(s) \) is formed by including in \( N_{AP}(s) \) the zeros in the right-half-plane and the poles at their mirrored locations in \( D_{AP}(s) \). \( H_{MP}(s) \) includes the rest of the poles and zeros. A 2nd-order delay equalizer is given by:

\[
\frac{V_o}{V_i} = \frac{s^2 - as + b}{s^2 + as + b}
\]

(2.6)

The complex poles and zeros of this function are symmetrical with respect to the \( j\omega \)-axis (Fig. 2.10).

In order to explain why this filter is referred as all-pass, consider that in \( H_{AP}(s) \), the roots of \( N_{AP}(s) \) are specular to the roots of \( D_{AP}(s) \), that are in the left-half-plane. Then,

\[
N_{AP}(s) = \pm D_{AP}(-s)
\]

(2.7)

Figure 2.10: Poles and zeros in a 2-nd order delay equalizer.

The magnitude is always \(|H_{AP}(j\omega)| = 1\). and thus,

\[
H_{AP}(j\omega) = e^{j\phi_{AP}(\omega)}
\]

(2.9)

where

\[
\phi_{AP}(\omega) = -2 \arctan \left( \frac{D_{AP}(-\omega)}{D_{AP}(\omega)} \right)
\]

(2.10)

An allpass network function can realize and arbitrary function of the phase without modifying the magnitude characteristic. In order to meet some particular magnitude and phase specifications, the first thing is finding an appropriate \( H_{AP}(s) \) to meet the magnitude requirements. After that an allpass function \( H_{AP}(s) \) is designed so as the total phase \( \phi_{AP}(\omega) \) or the total delay \( \tau_T(\omega) \) meets the phase or delay specifications. When both stages are cascaded phases, and delays, add up:

\[
\phi_T(\omega) = \phi(\omega) + \phi_{AP}(\omega)
\]

(2.11)

\[
\tau_T(\omega) = \tau(\omega) + \tau_{AP}(\omega)
\]

The total delay increases, but the important thing is that it can be made arbitrarily plain. The ideal filter has a linear phase or, equivalently, a constant delay.
2.3. Parameters normalization

An usual procedure in analog circuit design is normalizing both in frequency and impedance. Normalization does not lead to loss of generality. It is intended to make calculations easier. It helps in hand analysis, avoiding using large powers of 10. Normalization minimizes rounding error effects.

Frequency normalization consists in a change of scale. It is accomplished by dividing the frequency variable by a normalization frequency \( \Omega_o \). This frequency has to be properly selected. Then the normalized frequency variable will be:

\[
s_n = \frac{s}{\Omega_o}
\]  

(2.12)

Impedance level normalization is accomplished by dividing every impedance by a normalization resistance \( R_o \). Linear resistors, inductors and capacitors are transformed as follows

\[
R_n = \frac{R}{R_o} \\
L_n = L\Omega_o \frac{\Omega_o}{R_o} \\
C_n = C\Omega_o R_o
\]

(2.13)

yielding the following normalized values:

\[
R_n = \frac{R}{R_o} \\
L_n = L\Omega_o \frac{\Omega_o}{R_o} \\
C_n = C\Omega_o R_o
\]

(2.14)

2.4. Sensitivity

In the synthesis process, the circuit designer must select the best circuit configuration available. For this, the deviation of the real components from their nominal behaviour has to be considered. Deviations are due to:

a) fabrication process tolerances  
b) variations caused by environmental factors  
c) chemical changes related with aging  
d) the use of low accuracy models

For the implementation of a filter, all the coefficients of the network function \( H(s) \) depend on the elements of the circuit. Especially the location of the poles and zeros. A certain deviation of the filter operation from the ideal behaviour can be expected. The magnitude of the error will depend on the tolerances of the elements and the sensitivity of the circuit to each of these elements.

The concept of sensitivity is very useful when comparing different circuit configurations. It permits establishing their applicability to match some particular specifications. Sensitivity helps to:

a) select the best topology between all the available configurations  
b) know if the circuit will continue satisfying the specifications in the future.

For a given circuit element \( x \), any characteristic \( P \) of the circuit will depend on \( x \). If \( P \) is a network function, it will depend on the frequency as well: \( P = P(s, x) \). In order to find the deviation of \( P \) caused by an error \( dx = x - x_o \) of the element \( x \), we can perform a Taylor expansion of \( P(s, x) \) around \( x_o \):

\[
P(s, x) = P(s, x_o) + \frac{\partial P}{\partial x} P(s, x) \bigg|_{x_o} dx + \frac{1}{2} \frac{\partial^2 P}{\partial x^2} P(s, x) \bigg|_{x_o} (dx)^2 + \ldots
\]

(2.15)

Let us suppose that \( (dx/x_o) \ll 1 \) an that the curvature of \( P(s, x) \) around \( x_o \) is not large. Then 2nd derivatives and higher can be neglected:

\[
\Delta P(s, x_o) = P(s, x_o + dx) - P(s, x_o) \approx \frac{\partial P}{\partial x} P(s, x) \bigg|_{x_o} dx
\]

(2.16)

Considering the relative changes:

\[
\frac{\Delta P(s, x_o)}{P(s, x_o)} \approx \frac{x_o}{P(s, x_o) \partial x} P(s, x) \bigg|_{x_o} dx
\]

(2.17)

and then, the small signal sensitivity of \( P \) with respect to \( x \) is:

\[
S_x = \frac{\partial P}{\partial x} P(s, x) \bigg|_{x_o} = \frac{d lnP}{dx} \bigg|_{x_o}
\]

(2.18)
Also, the semirelative sensitivity can be defined as:

\[ Q_x^P = \frac{dP}{dx} \]  (2.19)

It can be useful in some cases, for instance, if \( S_x^P \) is evaluated where \( P = 0 \), then \( S_x^P \to \infty \), that does not report any information, while \( Q_x^P \) does. Also, sometimes absolute variations are more interesting than relative variations, like in the actual location of poles and zeros.

The relative change, variability, of a characteristic \( P \),

\[ \frac{\Delta P}{P} \equiv \frac{S_x^P}{x} \]  (2.20)

is \( S_x^P \) times the relative change in the parameter \( x \).

A good circuit should have small sensitivities. In these conditions, for an acceptable variability of \( P \), this is \( \Delta P/P \), elements with a larger tolerance, \( dx/x \), can be employed, resulting in a cheaper implementation. Also, in a circuit with small sensitivities, it is less probable that the components move out of the acceptability region during circuit operation because of their varying values.

These are the conclusions that can be extracted from this definition:

a) The Taylor expansion has been truncated at the second order term. This implies substituting \( P(x) \) by its slope at \( x = x_0 \). This means that the sensitivity \( S_x^P \) does not provide useful information if \( dx/x \) is not small.

b) If \( P \) depends on frequency, the sensitivity \( S_x^P \) is a function of frequency as well. The frequency ranges has to be considered too. For instance, the sensitivity of a network function to the gain of an amplifier in dc is useless to evaluate the operation at 10kHz.

c) \( P \) will depend on more than one parameter. It has been calculated for the nominal value of all of them. If any one changes, it has to be re-calculated. Free parameters are usually employed to reduce sensitivities.

d) At the end, the important parameter is the variability and not the sensitivity. A large sensitivity to very stable parameters can be acceptable, while a moderate sensitivity to elements with a large tolerance can render a design useless.

**Gain-sensitivity product**

The most employed active device in active filters design is the operational amplifier (OPAMP) (Fig. 2.11(a)). It is a high-gain differential voltage amplifier. Its output voltage is defined by:

\[ V_o = A(V_+ - V_-) \]  (2.21)

The open-loop gain, \( A \), is very large. It has also a large variability, \( dA/A \). Opamps are employed in feedback configuration with a reduced gain \( A \) but a smaller variability \( dA/A \). Two examples: the non-inverting and inverting amplifiers of Fig. 2.11(b) and Fig. 2.11(c). Consider the non-inverting amplifier of Fig. 2.11(b):

\[ V_2 = A \left( V_1 - \frac{R_1}{R_1 + (K-1)R_1} V_1 \right) = A \left( V_1 - \frac{1}{K} V_2 \right) \]  (2.22)

the closed-loop amplifier gain is:

\[ \mu_1 = \frac{V_2}{V_1} = \frac{K}{1 + K/A} \bigg|_{K \ll A} \cong K \]  (2.23)

Similarly, for Fig. 2.11(c) the gain obtained is:

\[ \mu_2 = \frac{V_2}{V_1} = \frac{K-1}{1 + K/A} \bigg|_{K \ll A} \cong -(K-1) \]  (2.24)

**Figure 2.11:** (a) OPAMP symbol, (b) amplifier with non-inverting feedback, (c) amplifier with inverting feedback.
The sensitivities of the closed-loop gain with respect to the gain in open-loop are:

\[ S_A^{\mu_1} = \frac{K/A}{1 + K/A} = \frac{\mu_1}{A} \]

\[ S_A^{\mu_2} = \frac{K/A}{1 + K/A} \approx \frac{\mu_2}{A} \quad \text{si} \quad K \gg 1 \]  

(2.25)

The sensitivity of \( \mu_i \) with respect to \( A \) increases with the closed-loop gain \( \mu_i \). Also the variability of \( \mu_i \) is more reduced that the variability of \( \mu \):

\[ \frac{d\mu_i}{\mu_i} = S_A^{\mu_i} \frac{dA}{A} \approx \frac{\mu_i dA}{A} \quad \text{(2.26)} \]

as it can be considered that \( \mu_i \ll A \).

For a certain filter characteristic \( P \), that depends on the closed-loop gain of an amplifier, we have:

\[ S_A^P = S_A^{\mu_i} \mu_i = \frac{\mu_i dA}{A} \]

(2.27)

and its variability will be:

\[ \frac{dP}{P} = (\mu_i S_A^{\mu_i} \frac{dA}{A})^2 = \Gamma_i \frac{dA}{A^2} \]

(2.28)

\( dP/P \) is proportional to the product of gain and sensitivity, \( \Gamma_i \), and to \( dA/A^2 \) that only depends on the amplifier. Then the variability of a certain characteristic does not only depend on the sensitivity, but also on the closed-loop gain that is required in the implementation. The gain-sensitivity product is a better parameter to establish a comparison between different designs.

For instance, consider an OPAMP with \( A = 10^3 \) and \( dA/A^2 = 60\% \). For a particular design, the sensitivity of a characteristic \( P \) is \( S_A^{\mu_1} = 6 \). In this design the closed-loop gain is \( \mu = 95 \). For a different design with the same functionality, the sensitivity is \( S_A^{\mu_2} = 38 \). This time the closed-loop gain required is only \( \mu = 4 \). The variabilities obtained are:

\[ \frac{dP_1}{P_1} = 95 \cdot 6 \cdot 0.6 \cdot 10^{-4} = 3.4\% \]

\[ \frac{dP_2}{P_2} = 4 \cdot 38 \cdot 0.6 \cdot 10^{-4} = 0.91\% \]

(2.29)

what means that the second design is better than the first, though it has a larger sensitivity.

Finally, it is interesting to know that the gain-sensitivity product is the same in both open and closed-loop configurations:

\[ \Gamma_A = \mu \]  

(2.30)

From (2.28) it can be extracted that the filter designer must select OPAMPs with the highest \( A \), in the frequency range of interest, in order to reduce \( dA/A^2 \).

Solved problems

2.1. Express the following function, that has non-minimum phase, as the product of a minimum phase function and an allpass function:

\[ H_{\text{NMP}}(s) = \frac{(s^2 + 2s + 6)(s^3 - 4s + 8)(s - 3)}{(s^2 + s + 4)(s^2 + 3s + 7)(s + 1)} \]

**Solution:**

\( H_{\text{NMP}}(s) \) has three zeros in the right-half-plane, one at \( s_1 = 3 \) and the other two at \( s_2, s_3 = 2 \pm j/2 \). The first step is adding three poles and three zeros to \( H_{\text{NMP}}(s) \), at the mirrored positions of \( s_1, s_2, \) and \( s_3 \):

1. In order to reach that equation, observe that:

\[ AS_A^P = A S_A^{\mu_1} S_A^{\mu_2} = \mu S_A^P \]
It can be split into these two parts:

\[
H_{AP}(s) = \frac{N_{AP}(s)}{D_{AP}(s)} = \frac{(s-3)(s^2-4s+8)}{(s+3)(s^2+4s+8)} \\
H_{MP}(s) = \frac{N_{MP}(s)}{D_{MP}(s)} = \frac{(s+3)(s^2+4s+8)(s^2+2s+6)}{(s+1)(s^2+s+4)(s^2+3s+7)}
\]  

(2.32)

where \(H_{NMP}(s) = H_{MP}(s)H_{AP}(s)\). The penalty is a higher order in the total system. It is increased in the number of right-half-plane zeros.

### 2.2. - The circuit in Fig. 2.12 implements a 2nd-order bandpass function with a pole frequency \(f_o = 3\text{kHz}\) and a quality factor \(Q = 20\). Find the corresponding values for the elements. Obtain the sensitivities \(\delta x\) and \(\delta K\), where \(x\) represents any passive element. \(K\) is the positive gain of a voltage amplifier. Ignore its limited bandwidth.

\[
\omega_o = \frac{1}{\sqrt{C^2R_1R_2}} \quad Q = \frac{r}{2 - r^2/(K-1)}
\]

(2.34)

where

\[
r = \frac{R_2}{\eta R_1}
\]

(2.35)

It can be noticed that \(\omega_o\) is not depending on the amplifier gain \(K\), then

\[
\delta K^0_o = 0
\]

(2.36)

which is very convenient. On the other side:

\[
\delta R_1^0_o = \frac{R_1}{\omega_o R_1} = \frac{1}{2} \quad \delta C^0_o = \frac{C}{\omega_o C} = -1
\]

(2.37)

This last one is \(-1\) because of the initial assumption: \(C_1 = C_2 = C\). If \(C_1 \neq C_2\) then

\[
\omega_o = \frac{1}{\sqrt{C_1C_2R_1R_2}}
\]

(2.38)
and, consequently
\[ S_{C_i}^{\omega_0} = -\frac{1}{2} \]  

(2.39)

It is interesting to notice that analyzing the dimensions of \( \omega_0 \), it has dimensions of \( \text{seconds}^{-1} \). It has to be inversely proportional to the square root of a product of time constants. Then, 0.5 is the minimum value that the sensitivity of \( \omega_0 \) with respect to any passive element of the filter can have. One of the design goals has to be yielding the minimum sensitivity with respect to the passive elements. Another is to avoid any dependence on the active elements.

Let us consider now the quality factor. There are two parameters, \( r \) and \( K \) that can help achieving \( Q = 20 \). They can be arbitrarily assigned, as long as \( K > 0.5r^2 + 1 \), because \( Q \) must be positive. If we choose \( r = 1 \) for convenience (equal resistors) the following results:

\[ Q = \frac{K - 1}{2K - 3} \]  

(2.40)

and if \( Q = 20 \) then \( K = K_o = 1.5128 \). This makes the sensitivity of \( Q \) with respect to \( K \) to be:

\[ S_K^Q = \frac{K_o Q_o}{Q_o \frac{\partial Q}{\partial K}} \bigg|_{K_o} = -115 \]  

(2.41)

then a change in the amplifier gain of a 0.25% triggers a change in \( Q \) of a 28%. As a 0.25% tolerance is rarely achieved in active devices, this circuit is useless. This is due to the abrupt slope of \( Q \) in the proximities of the pole at \( K = 1.5 \). Observe that a change of a 0.85% in \( K_o \) produces oscillations, because \( Q \) becomes negative for \( K < 1.5 \).

Moreover, recalling what was said in paragraph (a), a 0.5% change is not a small change. In this case (2.41) predicts a variation of ±57% in \( Q \) while applying (2.40) yields errors as large as ±36% and 140%.

Before rejecting this configuration, let us remember paragraph (c). It says that sensitivity is a function of all the parameters, and that free parameters can be employed to minimize sensitivities. Leaving \( r \) as a free parameter, let us compute the sensitivity of \( Q \) again:

\[ S_K^Q = -rQ \frac{K}{(K - 1)^2} \]  

(2.42)

Extracting \( 1/(K - 1) \) from (2.34) and substituting in (2.42):

\[ S_K^Q = 1 + \frac{4}{r^2} - \frac{1}{rQ} - \frac{2Q}{r^2} \left( 1 + \frac{2}{r} \right) \]  

(2.43)

Fig. 2.13 shows a representation of the absolute value of the sensitivity as a function of \( r \). Observe that \( r = 1 \) was an unfortunate selection. Sensitivity decreases with increasing \( r \)’s. A better selection is a larger \( r \), e.g. \( r = 6 \) that leads to \( S_K^Q = -5.9 \). This is a great enhancement in sensitivity, obtained at the expense of a larger range of values for the resistances.

Figure 2.13: Sensitivity as a function of \( r \).

The sensitivity of \( Q \) decreases with increasing \( r \)’s too:

\[ S_r^Q = -4Q \frac{1}{r} - 1 \]  

(2.44)

Therefore, the circuit of Fig. 2.12 is good if the selection of the elements is correct. Considering \( r = 6 \) and choosing \( C = 5nF \) these results follow:
The amplifier gain has increased but is still easily realizable.

2.3.- Consider the circuit of the previous exercise. The amplifier will be implemented with the non-inverting configuration of Fig. 2.14.

Suppose that an OPAMP of open-loop dc gain and variability is employed. Obtain the variability of the quality factor for both values of 1 and 6.

Solution:

From Fig. 2.14 the closed-loop gain obtained is:

\[ R_1 = \frac{1}{12\pi f_o C} = 1.768 \Omega \]

\[ R_2 = r^2 R_1 = 63.66 \Omega \quad K = 22.18 \]

\[ (2.45) \]

The amplifier gain has increased but is still easily realizable.

Although \( S_K^Q \) has been reduced from \(-115\) to \(-5.9\), \( \Delta Q / Q \) does not improve too much. The gain-sensitivity product results:

\[ r^Q = r Q \left( 1 + \frac{2}{r^2} - \frac{1}{r Q} \right)^2 \]

which has the minimum around \( r^2 = 6 \).

Proposed problems

2.4.- a) Obtain the sensitivity of the network function \( V_o(s)/V_i(s) \) in the circuit of the figure with respect to each of the elements of the circuit.

b) Obtain the sensitivities of \( \omega_0 \) and \( Q \) with respect to \( R, L \) and \( C \).

c) Obtain the variability of \( \omega_0 \) for a

i) +5% change in \( C \).

ii) +5% change in \( R \).
Appendix 2.1: Sensitivity of the network function

The network function of a filter can be expressed as:

\[ H(s) = \frac{N(s)}{D(s)} = \frac{\sum_i a_i s^i}{\sum_i b_i s^i} \]  

(2.51)

If the coefficients depend on element \( x \), the sensitivity of \( H(s) \) with respect to \( x \) is:

\[ S_x^H = S_x^N - S_x^D = x \left( \frac{1}{N} \frac{\partial N}{\partial x} - \frac{1}{D} \frac{\partial D}{\partial x} \right) \]  

(2.52)

As a function of the coefficients, it can be written as:

\[ S_x^H = x \left( \frac{\sum_i a_i s^i}{D} - \frac{\sum_i b_i s^i}{N} \right) \]  

(2.53)

It is interesting to obtain the sensitivity for a network function expressed in magnitude and phase:

\[ H(j\omega, x) = |H(j\omega, x)| e^{j\phi(\omega, x)} \]  

(2.54)

thus:

\[ S_x^{H(j\omega)} = S_x^{H(j\omega)} + jQ_x^{\phi(\omega)} \]  

(2.55)

The real part of the sensitivity of the network function corresponds to the sensitivity of the magnitude, while the imaginary part corresponds to the semirelative sensitivity of the phase.

The variation in the coefficients leads to shifts in the position of the poles and zeros. It is interesting to study the effect of these changes in \( H(s) \). Taking the logarithm of the network function expressed in terms of its zeros and poles:

\[ \ln H = \ln K + \sum_{i=1}^m \ln(s - z_i) - \sum_{i=1}^n \ln(s - p_i) \]  

(2.56)

where \( K = a_m / b_n \). Computing the derivatives and multiplying by \( x \):

\[ S_x^H = S_x^K - \sum_{i=1}^m Q_x^{z_i} + \sum_{i=1}^n Q_x^{p_i} \]  

(2.57)

As it is expected, these shifts are better perceived at the proximities of the poles and zeros locations. The sensitivity of the network function will be high in the vicinity of the poles and zeros. In sinusoidal steady state \( s = j\omega \), the network function sensitivity will tend to \( \infty \) at the transmission zeros, that lie at the \( j\omega \)-axis, and at poles with high \( Q \).

It can be observed also that \( S_x^H \) has poles in all the poles and zeros of \( H(s) \). Equation (2.57) is a partial fraction expansion of \( S_x^H \). The semirelatives sensitivities of the poles and zeros are \( Q_x^{z_i} \) and \( -Q_x^{p_i} \) the residues of the poles \( p_i \) and zeros \( z_i \) of \( S_x^H \).